Phys 410
Fall 2015
Lecture #18 Summary
29 October, 2015

To find $\frac{d\sigma}{d\Omega}(\theta,\varphi)$ we compare the area covered by the incident particles at impact parameters between b and $b+db$ (i.e. $d\sigma = 2\pi b db$) to the solid angle subtended by the exiting beam of particles (i.e. $d\Omega = 2\pi \sin\theta \, d\theta$) to arrive at $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$. To find the DSCS, we need to calculate the trajectory of a projectile particle for every possible impact parameter. We then did the example of a point particle elastically scattering from a fixed hard sphere of radius R in three dimensions and found that $\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$, which is independent of angle! The total scattering cross section is just $\sigma = \pi R^2$, which is the cross-sectional area presented by the sphere.

We considered Rutherford scattering and calculated the differential scattering cross section for scattering of an alpha particle (charge q) from a Au nucleus (charge Q). The two particles interact by means of the Coulomb force, which is parameterized as $F = \gamma/r^2$, with $\gamma = qQ/4\pi\varepsilon_0$. The orbit is a hyperbola, characteristic of an orbit of two particles interacting by means of a central inverse-square-law force with energy $E > 0$. From our previous work on the central-force two-body problem we know a lot about this orbit, and the fact that angular momentum is conserved, for example. By calculating the change in momentum of the alpha particle $|\Delta\vec{p}|$ in two ways, we found the relationship between the impact parameter and the scattering angle: $b = \frac{\gamma}{mv^2}\cot\theta/2$, where m is the alpha particle mass and v is its initial speed of the alpha when far from the nucleus. Putting this into the formula for the differential scattering cross section, $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$, we find: $\frac{d\sigma}{d\Omega} = \left(\frac{qQ/4\pi\varepsilon_0}{4E\sin^2(\theta/2)}\right)^2$.

This result was tested experimentally by Geiger and Marsden, who showed that the scattering rate scaled with n_{target} (by varying the thickness of the foil), scaled as $1/E^2$ (by varying the energy of the incident alpha particles) , scaled as $\frac{1}{\sin^4(\theta/2)}$ (by measuring the number of particles scattered vs. outgoing angle), and scaled as Z^2, where Q=+Ze is the nuclear charge by measuring scattering from Al, Cu, Ag, and Au.

Note that because $\frac{d\sigma}{d\Omega} \sim q^2Q^2$, the scattered particle distribution is insensitive to whether the Coulomb interaction is attractive or repulsive. Also, the agreement for the angular dependence of $\frac{d\sigma}{d\Omega}$ with data suggests that the Coulomb force has the simple $1/r^2$ dependence even down to nuclear length scales. This is a bit surprising since we are using classical physics to learn something about the sub-atomic domain. On the other hand, the Coulomb

interaction alone tells you a lot about the structure of the atom in quantum mechanics. Finally, the total scattering cross section calculated from this $\frac{d\sigma}{d\Omega}$ diverges. This is because the bare Coulomb force is infinitely long ranged. In reality, the Coulomb force of the nucleus is screened out by the electron cloud of the atom, on the length scale of one nm, or less. When this screening is taken into account the total scattering cross section becomes finite, as observed.

These calculations assume that the alpha particle only undergoes one scattering event in the material (the Born scattering approximation). In addition, because of the electron screening, when an alpha particle is near one nucleus, it is insensitive to all the other nuclei because they are 'cloaked' by their neutralizing electron clouds.

Up to this point we have considered Newtonian dynamics and Lagrangian dynamics. Now we consider Hamiltonian dynamics. The Lagrangian is written in terms of n generalized coordinates and their time derivatives. This set of parameters constitutes a $2n-$ dimensional **state space**. The Hamiltonian is written in terms of the generalized coordinates and their conjugate momenta, defined as $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. This set of $2n$ parameters constitutes **phase space**.

Recall from Lecture 15 that the Hamiltonian was derived to be $\mathcal{H} = \sum_{i=1}^{n} p_i\,\dot{q}_i - \mathcal{L}$, where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, and it is assumed that the Lagrangian has no explicit time dependence.

One can solve the n canonical momentum equations for $\dot{q}_i$ in terms of the coordinates q_i and momenta p_i to re-express all of the velocities in terms of just the coordinates and momenta $\dot{q}_i = \dot{q}_i(q_i, p_i)$. With this, one can express the Hamiltonian in terms of coordinates and momenta alone $\mathcal{H}(q,p)$, essentially employing a Legendre transformation to move from $(q_i,\ \dot{q}_i)$ to (q_i, p_i) as the independent variables. Taking the derivative of the Hamiltonian with respect to q_i and p_i, one finds Hamilton's equations: $\dot{q}_i = \partial\mathcal{H}/\partial p_i$ and $\dot{p}_i = -\partial\mathcal{H}/\partial q_i$, $i = 1, \ldots, n$. This is a set of $2n$ first-order differential equations, as opposed to the set of n second-order differential equations one gets from Lagrange's equations.

The Hamiltonian dynamics formulation is useful for quantum mechanics and for classical statistical mechanics. As a way of solving classical mechanics problems it has few advantages over Lagrangian dynamics.